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Slow Rates of Mixing for Dynamical Systems with Hyperbolic Structures

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Abstract We consider invertible discrete-time dynamical systems having a hyperbolic product structure in some region of the phase space with infinitely many branches and variable return time. We show that the decay of correlations of the SRB measure associated to that hyperbolic structure is related to the tail of the recurrence times. We also give sufficient conditions for the validity of the Central Limit Theorem. This extends previous results by Young in (Ann. Math. 147: 585–650, 1998; Israel J. Math. 110: 153–188, 1999).

Keywords Recurrence times · Decay of correlations · Central limit theorem

1 Introduction

One of the most powerful ways of describing the dynamical features of chaotic dynamical systems is through *invariant probability measures*, meaning that the probability of finding an orbit in a certain region of the phase space does not depend on the moment we consider. A map f is said to be *mixing* with respect to an invariant probability measure μ if

$$|\mu(f^{-n}(A)\cap B)-\mu(A)\mu(B)|\to 0, \quad \text{when } n\to\infty,$$

for any measurable sets A, B. Standard counterexamples show that in general there is no specific rate at which this convergence to zero occurs. However, defining the *correlation*

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function of *observables* $\varphi, \psi : M \to \mathbb{R}$,

$$\mathcal{C}_n(\varphi, \psi; \mu) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|,$$

it is sometimes possible to obtain specific rates of decay, which depend only on the map f (up to a multiplicative constant which is allowed to depend on φ and ψ), provided the observables φ and ψ have sufficient regularity. Notice that choosing these observables to be characteristic functions this gives exactly the definition of mixing. Still in this direction, the *Central Limit Theorem* states that the probability of a given deviation of the time average values from the spatial average is essentially given by a Normal Distribution.

Since the work of Bowen, Ruelle and Sinai [3, 7, 8] it is known that uniformly hyperbolic diffeomorphisms (Axiom A, Anosov) possess *SRB* (*or physical*) *measures* with exponential decay of correlations and satisfying the Central Limit Theorem. A key ingredient in the results of Sinai, Ruelle and Bowen are Markov partitions, which permit to deduce many statistical properties of the dynamical system through a codification of the dynamics.

In the context of non-uniformly hyperbolic diffeomorphisms, Young considered in [10] some Markov structures, in certain regions of the phase space, with infinitely many branches and variable return times. This structures played a key role in obtaining exponential decay of correlations and deduce the Central Limit Theorem for some classes of non-uniformly hyperbolic diffeomorphisms, including billiards with convex scatterers and Axiom A attractors. Still in this context we refer the work of Benedicks and Young in [2] where they also get exponential decay of correlations and deduce the Central Limit Theorem for Hénon maps. This approach has also been successfully implemented by Young in [11] for studying other rates of mixing of non-invertible dynamical systems.

The framework developed by Young in [10, 11] is certainly among the most powerful tools for studying the statistical properties of non-uniformly hyperbolic dynamical systems. In both approaches, there is an explicit relation between the tail of the recurrence times to the hyperbolic structure and the decay of correlations, at least for some specific rates. However, the results in both papers do not depict the whole scenario: on the one hand, the model in [10] can only be applied to systems whose decay of correlations is exponential; on the other hand, the model in [11], in spite of being suitable for other decay rates, is specific to non-invertible systems. A simple diffeomorphism as the *solenoid with intermittency* that we present in Sect. 2.4 does not fit the model in [10]; see Remark 2.4.

A reduction of the diffeomorphism case to an endomorphism has been successfully implemented in [10] for systems with exponential decay of return time. This reduction has not been carried out for other decays. Here we mix techniques from [10, 11] and fill in this gap. Hyperbolic structures with subexponential tail of recurrence times will certainly play an important role in obtaining the rates of mixing for the diffeomorphisms introduced by Viana in [9]. Such hyperbolic structures can also be useful in the study of some classes of billiards and Poincaré return maps for flows, for which the tails of recurrence frequently decay at subexponential rates.

Overview This work is organized in the following way. In Sect. 2 we present our main results. For that we present the hyperbolic structures that appeared in [2, 10] and introduce a diameter control on certain iterates of the elements in the hyperbolic structures. This diameter control plays a crucial role in our subexponential results. In Sect. 3 we consider an induced scheme with a tower extension and reduce the problem to the non-invertible case.

For this we need a careful control on the constants of the main results in [11]; we do this in Appendix A for the sake of completeness. We derive the results for the original dynamics in Sect. 4. Finally, a solenoid with intermittency is presented as an illustrating example in Sect. 5.

2 Statement of Results

2.1 Hyperbolic Structures

Consider $f\colon M \to M$, where M is a finite-dimensional Riemannian manifold, and let Leb be the *Lebesgue measure* on the Borel sets of M. Given a submanifold $\gamma \subset M$ we use Leb_γ to denote the measure on γ induced by the restriction of the Riemannian structure to γ.

An embedded disk $\gamma \subset M$ is called an *unstable manifold* if $\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$ for every $x, y \in \gamma$. Similarly, γ is called a *stable manifold* if $\mathrm{dist}(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to \infty$ for every $x, y \in \gamma$.

Definition 2.1 Let $\mathrm{Emb}^1(D^u, M)$ be the space of C^1 embeddings from a disk D^u into M. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s, a unit disk D^u of some $\mathbb{R}^n$, and a map $\Phi^u\colon K^s \times D^u \to M$ such that

(i) $\gamma^u = \Phi^u(\{x\} \times D^u)$ is an unstable manifold;
(ii) Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
(iii) $x \mapsto \Phi^u \mid (\{x\} \times D^u)$ defines a continuous map from K^s into $\mathrm{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined similarly.

Definition 2.2 We say that $\Lambda \subset M$ has a *hyperbolic product structure* if there exist a continuous family of unstable manifolds $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable manifolds $\Gamma^s = \{\gamma^s\}$ such that

(i) $\Lambda = (\bigcup \gamma^u) \cap (\bigcup \gamma^s)$;
(ii) $\dim \gamma^u + \dim \gamma^s = \dim M$;
(iii) each γ^s meets each γ^u in exactly one point;
(iv) stable and unstable manifolds are transversal with angles bounded away from 0.

Let $\Lambda \subset M$ have a hyperbolic product structure, whose defining families are Γ^s and Γ^u. A subset $\Lambda_0 \subset \Lambda$ is called an *s-subset* if Λ_0 also has a hyperbolic product structure and its defining families Γ_0^s and Γ_0^u can be chosen with $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined analogously. Given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$. For each $n \geq 1$ let $(f^n)^u$ denote the restriction of the map f^n to γ^u-disks, and let $\det D(f^n)^u$ be the Jacobian of $D(f^n)^u$. We require that the hyperbolic product structure Λ satisfies several properties:

($\mathbf{P}_1$) Markov: there are pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \ldots \subset \Lambda$ such that

(a) $\mathrm{Leb}_\gamma((\Lambda \setminus \bigcup \Lambda_i) \cap \gamma) = 0$ on each $\gamma \in \Gamma^u$;
(b) for each $i \in \mathbb{N}$ there is $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is u-subset, and for all $x \in \Lambda_i$

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$

In the statements of the remaining properties about the hyperbolic structure we assume that $C > 0$ and $0 < \beta < 1$ are constants which only depend on f and Λ.

(P$_2$) Contraction on stable leaves: $\mathrm{dist}(f^n(y), f^n(x)) \le C\beta^n$, $\forall y \in \gamma^s(x)$ $\forall n \ge 1$.

In spite of the uniform contraction in the stable direction, this condition is not too restrictive in systems having regions where the contraction fails to be uniform, since we are allowed to remove points in the unstable leaves, provided a subset with positive measure in those leaves remains at the end. This has been carried out in [2] for Hénon maps.

Next we introduce a return time function $R\colon \Lambda \to \mathbb{N}$ and a return map $f^R\colon \Lambda \to \Lambda$, defined for each $i \in \mathbb{N}$ as

$$R|_{\Lambda_i} = R_i \quad \text{and} \quad f^R|_{\Lambda_i} = f^{R_i}|_{\Lambda_i}.$$

We consider the *separation time* $s(x, y)$ for $x, y \in \Lambda$ as

$$s(x, y) = \min\big\{n \ge 0\colon (f^R)^n(x) \text{ and } (f^R)^n(y) \text{ lie in distinct } \Lambda_i\big\}.$$

The last two properties involve information on the action of f^R on unstable leaves.

(P$_3$) Regularity of the stable foliation: given $\gamma, \gamma' \in \Gamma^u$, we define $\Theta\colon \gamma' \cap \Lambda \to \gamma \cap \Lambda$ by $\Theta(x) = \gamma^s(x) \cap \gamma$. Then

(a) Θ is absolutely continuous and

$$\frac{d(\Theta_* \mathrm{Leb}_{\gamma'})}{d\,\mathrm{Leb}_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\Theta^{-1}(x)))};$$

(b) letting $u(x)$ denote the density in item (a), we have

$$\log \frac{u(x)}{u(y)} \le C\beta^{s(x,y)}, \quad \text{for } x, y \in \gamma' \cap \Lambda.$$

(P$_4$) Bounded distortion: for $\gamma \in \Gamma^u$ and $x, y \in \Lambda \cap \gamma$

$$\log \frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} \le C\beta^{s(f^R(x), f^R(y))}.$$

Remark 2.3 The Markov property we present here is weaker than the one in [10], since includes two extra assumptions: (i) there are at most finitely many i's with $R_i = n$ for each $n \in \mathbb{N}$; (ii) $R_i \ge R_0$ for some $R_0 > 1$ depending on the constants C and α. These assumptions play a role in showing the existence of a spectral gap for a transfer operator associated to the dynamics. Here we use a more probabilistic argument, based on [11], which enables us to drop those extra assumptions. In particular, we are able to reobtain the conclusions of [10] under our weaker Markov condition.

Remark 2.4 We do not assume any uniform backward contraction along unstable leaves similar to (P4) (a) in [10]. This would be too restrictive for our purposes, since the application we make of our main results does not have this property. Properties **(P$_3$)**(b) and **(P$_4$)** are new if comparing our setup to the one in [10]. However, they can be easily obtained from (P4) and (P5) in [10]; see [10, Lemma 1].

2.2 Diameter Control

Consider a sequence of *stopping times* defined for the points in Λ in the following way:

$$S_0 = 0, \qquad S_1 = R \quad \text{and} \quad S_{i+1} = S_i + R \circ f^{S_i}, \quad \text{for } i \geq 1. \tag{1}$$

We also define a nested sequence $(\mathcal{P}_k)_{k\geq 0}$ of partitions of Λ. Let $\mathcal{P}_0$ be the partition of Λ into the subsets Λ_i. Given $k \geq 1$, we say that x and y belong to an element of $\mathcal{P}_k$, if both conditions hold:

(i) $f^R(x)$ and $f^R(y)$ have the same stopping times $S_1 < \cdots < S_j$ up to time $k-1$;
(ii) $f^{S_i}(f^R(x))$ and $f^{S_i}(f^R(y))$ belong to the same element of $\mathcal{P}_0$ for each $0 \leq i \leq j$.

By construction we have that $S_{j+1}(f^R(x)) = S_{j+1}(f^R(y)) \geq k$ and $f^{S_{j+1}}(f^R(Q))$ is a u-subset.

We shall need to have a control on the diameter of certain iterates of the elements in the partitions defined above; see the proof of Lemma 3.2. Take any $k \geq 1$ and $P \in \mathcal{P}_0$. We consider separately the cases where k is bigger than $R(P)-1$ or not. If $k > R(P)-1$, then we define

$$\delta_k(P) = \sup_{0\leq \ell \leq R(P)-1} \left\{\operatorname{diam}(f^\ell(Q\cap\gamma))\colon \gamma\in\Gamma^u,\, Q\in\mathcal{P}_{k-R(P)+1+\ell},\, Q\subset P\right\}.$$

On the other hand, if $k \leq R(P)-1$, then we define the quantities

$$\delta_k^0(P) = \sup_{0\leq \ell < R(P)-k} \left\{\operatorname{diam}(f^\ell(P\cap\gamma))\colon \gamma\in\Gamma^u\right\},$$

$$\delta_k^+(P) = \sup_{R(P)-k\leq \ell \leq R(P)-1} \left\{\operatorname{diam}(f^\ell(Q\cap\gamma))\colon \gamma\in\Gamma^u,\, Q\in\mathcal{P}_{k-R(P)+1+\ell},\, Q\subset P\right\},$$

and

$$\delta_k(P) = \sup\{\delta_k^0(P), \delta_k^+(P)\}.$$

Finally we define

$$\delta_k = \sup_{P\in\mathcal{P}_0} \delta_k(P). \tag{2}$$

Though the definition of δ_k might seem somewhat technical, this is not so hard to calculate in practice, at least for some examples. One we have in mind is the solenoid with intermittency that we present in Sect. 2.4, for which we show in Sect. 5.2 that δ_k decays polynomially fast with k.

Remark 2.5 The argument in Sect. 5.2 can easily be adapted to show that δ_k *decays exponentially fast with k, once we know that the diameter of the elements Λ_i decay exponentially fast with R_i*. This includes all the examples studied in [10], since property (P4)(a) in [10] gives the exponential decay for the diameters of the elements in the initial partition with respect to the return time.

Remark 2.6 In the light of Definition 2.6 in [1] one may say that δ_k *decays exponentially fast with k whenever the return time R_i is a* hyperbolic time *for the points in Λ_i with respect to the derivative restricted to the tangent direction of the leaves in Γ^u*; see [1, Lemma 2.7] and recall Remark 2.5.

2.3 Main Results

The first result we present here asserts the existence of SRB measures for systems having some hyperbolic structure, provided the return time is integrable with respect to the conditional of the Lebesgue measure on some local unstable leaf.

Definition 2.7 We say that an f-invariant probability measure μ is a *Sinai-Ruelle-Bowen (SRB) measure* if f has no zero Lyapunov exponents μ almost everywhere, and the conditional measures on local unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds.

The proof of the next result is quite standard and may be found in [10].

Theorem A *Assume that f has a hyperbolic structure Λ such that* $\mathrm{Leb}_\gamma(\Lambda\cap\gamma)>0$ *for some $\gamma\in\Gamma^u$. If R is integrable with respect to* Leb_γ, *then f has some SRB measure μ.*

The next result shows that the decay of correlations of the SRB measure μ given by Theorem A is related to the recurrence times of the hyperbolic structure. It has been established by Young in [10, Theorem 2] a version of this result for hyperbolic structures having exponential decay of return time. The method in [10] is based on the existence of a spectral gap for the transfer operator and cannot be applied in our situation. We define the space of Hölder continuous functions with exponent $\eta>0$

$$H_\eta=\bigl\{\varphi\colon M\to\mathbb{R}\mid \exists C>0 \text{ such that } |\varphi(x)-\varphi(y)|\le C\,\mathrm{dist}(x,y)^\eta,\ \forall x,y\in M\bigr\}.$$

Theorem B *Assume that f has a hyperbolic structure Λ for which* **(P$_1$)**–**(P$_4$)** *hold, with* $\gcd\{R_i\}=1$ *and* $\mathrm{Leb}_\gamma(\Lambda\cap\gamma)>0$ *for some $\gamma\in\Gamma^u$. Given $\varphi,\psi\in H_\eta$,*

(1) *if* $\mathrm{Leb}_\gamma\{R>n\}\lesssim n^{-\alpha}$ *for some $\alpha>1$, then* $\mathcal{C}_n(\varphi,\psi;\mu)\lesssim\max\{n^{-\alpha+1},\delta_n^\eta\}$;
(2) *if* $\mathrm{Leb}_\gamma\{R>n\}\lesssim e^{-cn^\zeta}$ *for some $c>0$ and $0<\zeta\le1$, then there exists $c'>0$ such that* $\mathcal{C}_n(\varphi,\psi;\mu)\lesssim\max\{e^{-c'n^\zeta},\delta_n^\eta\}$.

As shown in [10, Sect. 4.1], condition $\gcd\{R_i\}=1$ can be replaced by the assumption that f^n is ergodic with respect to μ for every $n\ge1$. If we omit both assumptions, then the same conclusion holds for some power of f. The next result gives the *Central Limit Theorem* for Hölder continuous observables which are not a coboundary with respect to the SRB measure μ.

Theorem C *Under the assumptions of Theorem B, if* $\mathrm{Leb}_\gamma\{R>n\}\lesssim n^{-\alpha}$ *for some $\alpha>2$, then given $\varphi\in H_\eta$ for which there is no $\psi\in L^2(\mu)$ with $\varphi=\psi\circ f-\psi$ there exists $\sigma>0$ such that for every interval $J\subset\mathbb{R}$,*

$$\mu\left\{x\in M:\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\left(\varphi(f^j(x))-\int\varphi\,d\mu\right)\in J\right\}\xrightarrow{n\to\infty}\frac{1}{\sigma\sqrt{2\pi}}\int_J e^{-t^2/2\sigma^2}dt.$$

2.4 Application

We give a diffeomorphism for which we may apply our main results and deduce that it has an SRB measure with polynomial decay of correlations. This is obtained by perturbing the classical solenoid map in the unstable direction of one fixed point and transforming it into an indifferent fixed point. Let $f: S^1 \to S^1$ be a map of degree $d \geq 2$ with the following properties:

(i) f is C^2 on $S^1 \setminus \{0\}$;
(ii) f is C^1 on S^1 and $f' > 1$ on $S^1 \setminus \{0\}$;
(iii) $f(0) = 0$, $f'(0) = 1$, and there is $\gamma > 0$ such that

$$-xf''(x) \approx |x|^\gamma \quad \text{for all } x \neq 0.$$

Consider the solid torus $M = S^1 \times D^2$, where D^2 is the unit disk in $\mathbb{R}^2$, and define the map $g: M \to M$ by

$$g(x, y, z) = \left(f(x), \frac{1}{10}y + \frac{1}{2}\cos x, \frac{1}{10}z + \frac{1}{2}\sin x \right).$$

Let H_η be the space of Hölder continuous functions on M with exponent $\eta > 0$.

Theorem D *Let $g: M \to M$ be as above and take $\varphi, \psi \in H_\eta$.*

(1) *The map g admits an SRB measure μ if and only if $\gamma < 1$.*
(2) *Assume that $\gamma < 1$. Then*
   (a) *for $\eta \geq 1 - \gamma$ we have $\mathcal{D}_n(\varphi, \psi; \mu) \lesssim n^{1-1/\gamma}$;*
   (b) *for $\eta < 1 - \gamma$ we have $\mathcal{D}_n(\varphi, \psi; \mu) \lesssim n^{-\eta/\gamma}$.*
(3) *If $\gamma < 1/2$, then the Central Limit Theorem holds for $\varphi \in H_\eta$, provided there is no $\psi \in L^2(\mu)$ with $\varphi = \psi \circ f - \psi$.*

It is well known that for $\gamma \geq 1$ one has $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$ converging in the weak* topology to the Dirac measure at 0 for Lebesgue almost every $x \in S^1$; see for example [5, 6]. Using the fact that we have uniform contraction in the vertical direction, it is not hard to see that $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{g^j(x,y)}$ converges in the weak* topology to the Dirac measure at 0 for Lebesgue almost every $(x, y) \in S^1 \times D^2$. This observation justifies the "only if" part of the theorem above.

3 Induced Schemes

Here we consider an induced scheme with a tower extension and reduce the problem to a non-invertible case. We follow closely the approach in [10].

3.1 The Natural Measure

Fix an arbitrary $\hat{\gamma} \in \Gamma^u$. Given $\gamma \in \Gamma^u$ and $x \in \gamma \cap \Lambda$ let $\hat{x}$ be the point in $\gamma^s(x) \cap \hat{\gamma}$. Defining for $x \in \gamma \cap \Lambda$

$$\hat{u}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\hat{x}))}$$

we have that $\hat{u}$ satisfies the bounded distortion property ($\mathbf{P}_3$)(b). For each $\gamma \in \Gamma^u$ let m_γ be the measure in γ such that

$$\frac{dm_\gamma}{d\,\mathrm{Leb}_\gamma} = \hat{u}\mathbf{1}_{\gamma\cap\Lambda},$$

where $\mathbf{1}_{\gamma\cap\Lambda}$ is the characteristic function of the set $\gamma\cap\Lambda$. These measures have been defined in such a way that if $\gamma,\gamma'\in\Gamma^u$ and Θ is obtained by sliding along stable leaves from $\gamma\cap\Lambda$ to $\gamma'\cap\Lambda$, then

$$\Theta_* m_\gamma = m_{\gamma'}. \tag{3}$$

To verify this let us show that the densities of these two measures with respect to Leb_γ coincide. Take $x\in\gamma\cap\Lambda$ and $x'\in\gamma'\cap\Lambda$ such that $\Theta(x)=x'$. By (**P**$_3$)(a) one has

$$\frac{d\Theta_*\,\mathrm{Leb}_\gamma}{d\,\mathrm{Leb}_{\gamma'}}(x') = \frac{\hat u(x')}{\hat u(x)},$$

which implies that

$$\frac{d\Theta_* m_\gamma}{d\,\mathrm{Leb}_{\gamma'}}(x') = \hat u(x)\frac{d\Theta_*\,\mathrm{Leb}_\gamma}{d\,\mathrm{Leb}_{\gamma'}}(x') = \hat u(x') = \frac{dm_{\gamma'}}{d\,\mathrm{Leb}_{\gamma'}}(x').$$

Lemma 3.1 *Assuming that $f^R(\gamma\cap\Lambda)\subset\gamma'$ for $\gamma,\gamma'\in\Gamma^u$, let $Jf^R(x)$ denote the Jacobian of f^R with respect to the measures m_γ and $m_{\gamma'}$. Then*

(1) *$Jf^R(x)=Jf^R(y)$ for every $y\in\gamma^s(x)$;*
(2) *there is $C_1>0$ such that for every $x,y\in\Lambda\cap\gamma$*

$$\left|\frac{Jf^R(x)}{Jf^R(y)}-1\right| \le C_1\beta^{s(f^R(x),f^R(y))}.$$

Proof (1) For Leb_γ almost every $x\in\gamma\cap\Lambda$ we have

$$Jf^R(x) = \left|\det D(f^R)^u(x)\right|\cdot\frac{\hat u(f^R(x))}{\hat u(x)}. \tag{4}$$

Denoting $\varphi(x)=\log|\det Df^u(x)|$ we may write

$$\begin{aligned}
\log Jf^R(x) &= \sum_{i=0}^{R-1}\varphi(f^i(x)) + \sum_{i=0}^{\infty}\Big(\varphi\big(f^i(f^R(x))\big)-\varphi\big(f^i(\widehat{f^R(x)})\big)\Big)\\
&\quad -\sum_{i=0}^{\infty}\big(\varphi(f^i(x))-\varphi(f^i(\hat x))\big)\\
&= \sum_{i=0}^{R-1}\varphi(f^i(\hat x)) + \sum_{i=0}^{\infty}\Big(\varphi\big(f^i(f^R(\hat x))\big)-\varphi\big(f^i(\widehat{f^R(x)})\big)\Big).
\end{aligned}$$

Thus we have shown that $Jf^R(x)$ can be expressed just in terms of $\hat x$ and $\widehat{f^R(x)}$, which is enough for proving the first part of the lemma.

(2) It follows from (4) that

$$\log\frac{Jf^R(x)}{Jf^R(y)} = \log\frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} + \log\frac{\hat u(f^R(x))}{\hat u(f^R(y))} + \log\frac{\hat u(y)}{\hat u(x)}.$$

Observing that $s(x,y)>s(f^R(x),f^R(y))$ the conclusion follows from (**P**$_3$)(b) and (**P**$_4$). □

3.2 A Tower Extension

We introduce a tower extension of the dynamical system f restricted to $\bigcup_{n\geq 0} f^n(\Lambda)$; note that this space is preserved by f. We define a *tower*

$$\Delta = \big\{(x,\ell)\colon x\in\Lambda \text{ and } 0\leq \ell < R(x)\big\},$$

and a *tower map* $F:\Delta\to\Delta$ as

$$F(x,\ell)=\begin{cases}(x,\ell+1), & \text{if } \ell+1<R(x);\\ (f^R(x),0), & \text{if } \ell+1=R(x).\end{cases}$$

The *ℓth level of the tower* is by definition the set

$$\Delta_\ell=\{(x,\ell)\in\Delta\}.$$

The 0th-level of the tower Δ_0 is naturally identified with Λ and we shall make no distinction between them. Under this identification it easily follows from the definitions that $F^R=f^R$ for each $x\in\Delta_0$. Note that the ℓth level of the tower is a copy of the set $\{R>\ell\}\subset\Delta_0$. Also, we easily obtain a partition $\mathcal{P}$ of Δ_0 into subsets $\Delta_{0,i}$, with $\Delta_{0,i}=\Lambda_i$ for $i\geq 1$. This partition gives rise to partitions $\Delta_{\ell,i}$ on each tower level ℓ, considering

$$\Delta_{\ell,i}=\{(x,\ell)\in\Delta_\ell :\ x\in\Delta_{0,i}\}.$$

Collecting all these sets we obtain a partition $\mathcal{Q}=\{\Delta_{\ell,i}\}_{\ell,i}$ of Δ. We introduce a sequence of partitions $(\mathcal{Q}_n)_{n\geq 0}$ of Δ in the following way:

$$\mathcal{Q}_0=\mathcal{Q},\quad \text{and}\quad \mathcal{Q}_n=\bigvee_{i=0}^{n} F^{-i}\mathcal{Q}\quad \text{for } n\geq 0. \tag{5}$$

We shall denote by $Q_n(x)$ the element in $\mathcal{Q}_n$ containing the point $x\in\Delta$.

We define a projection map

$$\begin{aligned}\pi\colon\quad \Delta &\longrightarrow \bigcup_{n\geq 0} f^n(\Delta_0),\\ (x,\ell) &\longmapsto f^\ell(x).\end{aligned} \tag{6}$$

Observe that $f\circ\pi=\pi\circ F$.

Lemma 3.2 *There is $C_2>0$ such that for all $k\geq 0$ and $Q\in\mathcal{Q}_{2k}$*

$$\operatorname{diam}(\pi F^k(Q))\leq C_2\max\{\beta^k,\delta_k\}.$$

Proof Take $k\geq 0$ and $Q\in\mathcal{Q}_{2k}$. Given $x,y\in Q$, there is $z\in\gamma^u(x)\cap\gamma^s(y)$. Supposing that $Q\subset\Delta_\ell$, then $y_0=\pi F^{-\ell}(y)$ and $z_0=\pi F^{-\ell}(z)$ are both in Δ_0 and they lie on the same stable leaf. Hence

$$\begin{aligned}\operatorname{dist}(\pi F^k(y), \pi F^k(z)) &= \operatorname{dist}(\pi F^{k+\ell}(y_0), \pi F^{k+\ell}(z_0))\\ &= \operatorname{dist}(f^{k+\ell}(\pi y_0), \pi f^{k+\ell}(\pi z_0)).\end{aligned}$$

Using $(\mathbf{P}_2)$ we get

$$\operatorname{dist}(\pi F^k(y), \pi F^k(z)) \le C\beta^{k+\ell}. \tag{7}$$

On the other hand, we have $F^k(Q) \in \mathcal{Q}_k$, which implies that $F^k(x)$ and $F^k(z)$ are both in an unstable leaf of some element of $\mathcal{Q}_k$. In particular, there are $P \in \mathcal{P}_0$ and $\ell < R(P)$ such that element of $\mathcal{Q}_k$ is in the ℓ-th level of the tower over P. Moreover, the situations considered for defining $\delta_k(P)$ correspond precisely to the possible cases for the elements of $\mathcal{Q}_k$ over P. Taking into account the definition of π, this gives

$$\operatorname{dist}(\pi F^k(x), \pi F^k(z)) \le \delta_k(P),$$

which together with (7) gives the desired conclusion. □

Let m be the measure on Λ whose conditional measures on $\gamma \cap \Lambda$ with $\gamma \in \Gamma^u$ are the measures m_γ introduced in the previous section. This measure m allows us to introduce a measure on Δ that we still denote m, by letting $m|\Delta_\ell$ be the measure induced by the natural identification of Δ_ℓ with a subset of Λ. We let JF denote the Jacobian of F with respect to this measure m.

Lemma 3.3 *There is $C_F > 0$ such that for all $k \ge 1$ and all $x, y \in \Delta$ belonging to a same element of $\mathcal{Q}_{k-1}$*

$$\left|\frac{JF^k(x)}{JF^k(y)} - 1\right| \le C_F \beta^{s(F^k(x), F^k(y))}.$$

Proof By Lemma 3.1 one knows that for all $i \ge 1$ and all $x, y \in \Delta_{0,i}$

$$\left|\frac{JF^R(x)}{JF^R(y)} - 1\right| \le C_1 \beta^{s(F^R(x), F^R(y))}. \tag{8}$$

It follows that there is a constant $C_F > 0$ such that for all $n \ge 1$ and all x, y belonging to a same element of $\vee_{j=0}^{n-1}(F^R)^{-j}\mathcal{P}$

$$\left|\frac{J(F^R)^n(x)}{J(F^R)^n(y)} - 1\right| \le C_F \beta^{s((F^R)^n(x), (F^R)^n(y))}. \tag{9}$$

In fact, if x and y belong to a same element of $\vee_{j=0}^{n-1}(F^R)^{-j}\mathcal{P}$, then $(F^R)^j(x)$ and $(F^R)^j(y)$ belong to a same element of $\mathcal{P}$ for every $0 \le j < n$. Moreover,

$$s\big((F^R)^j(x), (F^R)^j(y)\big) = s\big((F^R)^n(x), (F^R)^n(y)\big) + (n-j). \tag{10}$$

Then

$$\log \frac{J(F^R)^n(x)}{J(F^R)^n(y)} = \sum_{j=0}^{n-1} \log \frac{JF^R((F^R)^j(x))}{JF^R((F^R)^j(y))}$$

$$\leq \sum_{j=0}^{n-1} C_1 \beta^{s((F^R)^n(x),(F^R)^n(y))+(n-j)-1}, \quad \text{by (8) and (10)}$$

$$\leq C_F \beta^{s((F^R)^n(x),(F^R)^n(y))}, \tag{11}$$

where $C_F > 0$ depends only on C_1 and β. This implies that (9) holds.

From (9) we easily deduce that for all $k \geq 1$ and all $x, y \in \Delta$ belonging to a same element of $\mathcal{Q}_{k-1}$

$$\left| \frac{JF^k(x)}{JF^k(y)} - 1 \right| \leq C_F \beta^{s(F^k(x),F^k(y))}. \tag{12}$$

To see this, we consider $JF^k(x) = J(F^R)^n(x')$ and $JF^k(y) = J(F^R)^n(y')$, where n is the number of visits of x and y to Δ_0 prior to time k, and x', y' are the elements in the bottom level Δ_0 corresponding x, y, respectively. In this way, we have x', y' belonging to a same element of $\bigvee_{j=0}^{n-1}(F^R)^{-j}\mathcal{P}$ and $s(x, y) = s(x', y')$. Using (9) we obtain (12). □

3.3 Quotient Dynamics

Let $\bar{\Lambda} = \Lambda/\sim$, where $x \sim y$ if and only if $y \in \gamma^s(x)$. This quotient space gives rise to a quotient tower $\bar{\Delta}$ with levels $\bar{\Delta}_\ell = \Delta_\ell/\sim$. A partition of $\bar{\Delta}$ into $\bar{\Delta}_{0,i}$, that we denote by $\bar{\mathcal{P}}$, and a sequence $\bar{\mathcal{Q}}_n$ of partitions of $\bar{\Delta}$ as in (5) are defined in a natural way.

As f^R takes γ^s-leaves to γ^s-leaves and R has been defined in such a way that it does not depend on the point we take in a same stable leaf, we may assume that we have defined the return time $\bar{R}\colon \bar{\Delta}_0 \to \mathbb{N}$, the tower map $\bar{F}\colon \bar{\Delta} \to \bar{\Delta}$ and the separation time $\bar{s}\colon \bar{\Delta}_0 \times \bar{\Delta}_0 \to \mathbb{N}$ naturally induced by the corresponding ones in Δ_0 and Δ. It will be convenient to have this separation time defined in the whole $\bar{\Delta}$. This may be done by taking $\bar{s}(x, y) = \bar{s}(x', y')$ if x and y belong in a same $\bar{\Delta}_{l,i}$, where x', y' are the corresponding elements of $\bar{\Delta}_{0,i}$, and $\bar{s}(x, y) = 0$ otherwise.

Since (3) holds, we may introduce a measure $\bar{m}$ on $\bar{\Delta}$ whose representative on each $\gamma \in \Gamma^u$ is m_γ. We let $J\bar{F}$ denote the Jacobian of $\bar{F}$ with respect to this measure $\bar{m}$. The first item of Lemma 3.1 shows that the Jacobian $J\bar{F}$ is well defined with respect to $\bar{m}$. From Lemma 3.3 we easily obtain:

Lemma 3.4 *For all $k \geq 1$ and all $x, y \in \bar{\Delta}$ belonging to a same element of $\bar{\mathcal{Q}}_{k-1}$*

$$\left| \frac{J\bar{F}^k(x)}{J\bar{F}^k(y)} - 1 \right| \leq C_F \beta^{\bar{s}(\bar{F}^k(x),\bar{F}^k(y))}.$$

It will be useful to consider $\hat{R}\colon \bar{\Delta} \longrightarrow \mathbb{N}$ defined as

$$\hat{R}(x) = \min\{n \geq 0\colon\ \bar{F}^n(x) \in \bar{\Delta}_0\}.$$

Note that $\hat{R}(x) = \bar{R}(x)$ for all $x \in \bar{\Delta}_0$, and

$$\bar{m}\{\hat{R} > n\} = \sum_{l>n} \bar{m}(\bar{\Delta}_l) = \sum_{l>n} \bar{m}\{\bar{R} > l\}.$$

We introduce the spaces of Hölder functions in $\bar{\Delta}$

$$\mathcal{F}_\beta = \left\{\varphi : \bar{\Delta} \to \mathbb{R} \mid\ \exists C_\varphi > 0 \text{ such that } |\varphi(x) - \varphi(y)| \leq C_\varphi \beta^{\bar{s}(x,y)} \text{ for all } x, y \in \bar{\Delta}\right\}$$

$$\mathcal{F}_\beta^+ = \Big\{ \varphi \in \mathcal{F}_\beta \mid \exists C_\varphi > 0 \text{ such that on each } \bar{\Delta}_{\ell,i}, \text{ either } \varphi \equiv 0, \text{ or}$$

$$\varphi > 0 \text{ and } \left| \frac{\varphi(x)}{\varphi(y)} - 1 \right| \le C_\varphi \beta^{\bar{s}(x,y)} \text{ for all } x, y \in \bar{\Delta}_{\ell,i} \Big\}.$$

The following result gives the existence of an equilibrium measure for the tower map and some of its properties. A proof of it is given in [10, Lemma 2] and [11, Theorem 1].

Theorem 3.5 *Assume that $\bar{R}$ is integrable with respect to $\bar{m}$. Then*

(1) *$\bar{F}$ has a unique absolutely continuous invariant probability $\bar{\nu}$ equivalent to $\bar{m}$;*
(2) *$d\bar{\nu}/d\bar{m}$ belongs to $\mathcal{F}_\beta^+$ and is bounded from below by some $c > 0$;*
(3) *$(\bar{F}, \bar{\nu})$ is exact and, hence ergodic and mixing.*

The decay of correlations for the measure $\bar{\nu}$ has been proved in [10]. This occurs at the same speed that the positive iterates under $\bar{F}_*$ of measures with densities in $\mathcal{F}_\beta^+$ converge to the equilibrium $\bar{\nu}$. This speed is related to the decay of $\bar{m}\{\bar{R} > n\}$, at least for some specific rates.

Theorem 3.6 *For $\varphi \in \mathcal{F}_\beta^+$ let $\bar{\lambda}$ be the measure whose density with respect to $\bar{m}$ is φ.*

(1) *If $\bar{m}\{\bar{R} > n\} \le Cn^{-\zeta}$, for some $C > 0$ and $\zeta > 1$, then there is $C' > 0$ such that*

$$\left| \bar{F}_*^n \bar{\lambda} - \bar{\nu} \right| \le C' n^{-\zeta+1}.$$

(2) *If $\bar{m}\{\bar{R} > n\} \le Ce^{-cn^\eta}$, for some $C, c > 0$ and $0 < \eta \le 1$, then there are $C', c' > 0$ such that*

$$\left| \bar{F}_*^n \bar{\lambda} - \bar{\nu} \right| \le C' e^{-c' n^\eta}.$$

Moreover, c' does not depend on φ and C' depends only on C_φ.

A version of this theorem has been proved in [10, Theorem 2] but without establishing the dependence on the constants. This plays a crucial role in our proofs of Theorem B and Theorem C. Following the arguments in [10] we give a detailed proof of Theorem 3.6 in Appendix A.

4 Back to the Original Dynamics

Let π be the map from Δ to M defined in (6). Let also $\bar{\pi}$ be the projection from Δ to the quotient space $\bar{\Delta}$. As observed in [10, Sects. 2 and 4] we have $\bar{\nu} = \bar{\pi}_* \nu$ and $\mu = \pi_* \nu$. Given $\varphi, \psi \in H_\eta$ we define $\tilde{\psi} = \psi \circ \pi$ and $\tilde{\varphi} = \varphi \circ \pi$.

4.1 Decay of Correlations

For proving Theorem B we start by noting that for $\varphi, \psi \in H_\eta$ we have

$$\int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu = \int (\tilde{\varphi} \circ F^n) \tilde{\psi} d\nu - \int \tilde{\varphi} d\nu \int \tilde{\psi} d\nu,$$

which shows that it suffices to obtain the desired conclusions for $\mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}; \nu)$. This will be done in several steps, firstly reducing it to a problem in $\bar{\Delta}$ and then applying Theorem 3.6.

Step 1 Fix some positive integer $k \le n/4$. Consider a discretization $\bar{\varphi}_k$ of $\tilde{\varphi}$ defined on Δ (or $\bar{\Delta}$) as

$$\bar{\varphi}_k|_A = \inf\{\tilde{\varphi} \circ F^k(x) \colon x \in A\}, \quad \text{for } A \in \mathcal{Q}_{2k}.$$

We have

$$|\mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}; \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}; \nu)| \le C_3 \delta_k^\eta \tag{13}$$

for some C_3 depending only on C_φ and $\|\psi\|_\infty$.

Actually, by Lemma 3.2 one knows that $|\tilde{\varphi} \circ F^k - \bar{\varphi}_k| \le C_\varphi (C_2 \delta_k)^\eta$. To be precise, one should consider the case $\beta^k > \delta_k$, but this would only be relevant in the second part of Theorem B. However, it does not play any special role for the conclusion.

Observing that $\mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}; \nu) = \mathcal{C}_{n-k}(\tilde{\varphi} \circ F^k, \tilde{\psi}; \nu)$, the left-hand side of inequality (13) is

$$\begin{aligned}
&\le \left| \int (\tilde{\varphi} \circ F^k - \bar{\varphi}_k) \circ F^{n-k} \cdot \tilde{\psi}\, d\nu \right| + \left| \int (\tilde{\varphi} \circ F^k - \bar{\varphi}_k) d\nu \cdot \int \tilde{\psi}\, d\nu \right| \\
&\le 2C_\varphi (C_2 \delta_k)^\eta \|\psi\|_\infty .
\end{aligned}$$

We just have to take $C_3 = 2C_\varphi C_2^\eta \|\psi\|_\infty$.

Step 2 Consider $\bar{\psi}_k$ defined similarly to $\bar{\varphi}_k$ above. Let $\bar{\psi}_k \nu$ denote the signed measure whose density with respect to ν is $\bar{\psi}_k$, and let $\tilde{\psi}_k$ denote the density of $F_*^k(\bar{\psi}_k \nu)$ with respect to ν. Then

$$|\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}; \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}_k; \nu)| \le C_4 \delta_k^\eta, \tag{14}$$

for some C_4 depending only on C_ψ and $\|\varphi\|_\infty$.

In fact, the left-hand side of (14) is

$$\begin{aligned}
&\le \left| \int (\bar{\varphi}_k \circ F^{n-k})(\tilde{\psi} - \tilde{\psi}_k) d\nu \right| + \left| \int \bar{\varphi}_k d\nu \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right| \\
&\le 2\|\varphi\|_\infty \cdot \left| \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right| .
\end{aligned}$$

Letting $|\cdot|$ denote the total variation of a signed measure, and noting that

$$F_*^k((\tilde{\psi} \circ F^k)\nu) = \tilde{\psi}\nu,$$

we have

$$\begin{aligned}
\left| \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right| &= |\tilde{\psi}\nu - \tilde{\psi}_k \nu| = |F_*^k((\tilde{\psi} \circ F^k)\nu) - F_*^k(\bar{\psi}_k \nu)| \\
&\le |(\tilde{\psi} \circ F^k - \bar{\psi}_k)\nu| = \int |\tilde{\psi} \circ F^k - \bar{\psi}_k| d\nu.
\end{aligned}$$

By Lemma 3.2 one has $|\tilde{\psi} \circ F^k - \bar{\psi}_k| \le C_\psi (C_2 \delta_k)^\eta$. Take $C_4 = 2C_\psi C_2^\eta \|\varphi\|_\infty$.

Step 3 Now we show that

$$\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}_k; \nu) = \mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k; \bar{\nu}). \tag{15}$$

Indeed,

$$\int (\bar{\varphi}_k \circ F^{n-k})\tilde{\psi}_k d\nu = \int \bar{\varphi}_k d(F_*^{n-k}(\tilde{\psi}_k \nu)) = \int \bar{\varphi}_k d(F_*^n(\bar{\psi}_k \nu)),$$

and since $\bar{\varphi}_k$ is constant on γ^s leaves and F and $\bar{F}$ are semi-conjugated by $\bar{\pi}$, we have

$$\int \bar{\varphi}_k d(F_*^n(\bar{\psi}_k \nu)) = \int \bar{\varphi}_k d(\bar{\pi}_* F_*^n(\bar{\psi}_k \nu)) = \int \bar{\varphi}_k d(\bar{F}_*^n(\bar{\psi}_k \bar{\nu})) = \int (\bar{\varphi}_k \circ F^n)\bar{\psi}_k d\bar{\nu}.$$

Thus we have proved that

$$\int (\bar{\varphi}_k \circ F^{n-k})\tilde{\psi}_k d\nu = \int (\bar{\varphi}_k \circ F^n)\bar{\psi}_k d\bar{\nu}.$$

On the other hand,

$$\int \bar{\varphi}_k d\nu \cdot \int \tilde{\psi}_k d\nu = \int \bar{\varphi}_k d\bar{\nu} \cdot \int d(F_*^k(\bar{\psi}_k \nu)) = \int \bar{\varphi}_k d\bar{\nu} \cdot \int \bar{\psi}_k d\bar{\nu}.$$

These last to formulas give precisely (15).

Step 4 With no loss of generality we assume that $\bar{\psi}_k$ is not the null function. Taking

$$b_k = \left(\int (\bar{\psi}_k + 2\|\bar{\psi}_k\|_\infty) d\bar{\nu}\right)^{-1} \quad \text{and} \quad \hat{\psi}_k = b_k(\bar{\psi}_k + 2\|\bar{\psi}_k\|_\infty),$$

we then have

$$\int \hat{\psi}_k \bar{\rho} d\bar{m} = 1, \quad \text{where } \bar{\rho} = \frac{d\bar{\nu}}{d\bar{m}}.$$

Moreover,

$$\frac{1}{3\|\bar{\psi}_k\|_\infty} \le b_k \le \frac{1}{\|\bar{\psi}_k\|_\infty} \quad \text{and} \quad 1 \le \|\hat{\psi}_k\|_\infty \le 3.$$

Observe that $\hat{\psi}_k$ is constant on elements of $\mathcal{Q}_{2k}$, since $\bar{\psi}_k$ has this property. Let $\hat{\lambda}_k$ be the probability measure on $\bar{\Delta}$ whose density with respect to $\bar{m}$ is $\hat{\psi}_k \bar{\rho}$. Then,

$$\begin{aligned}\left|\int (\bar{\varphi}_k \circ \bar{F}^n)\bar{\psi}_k d\bar{\nu} - \int \bar{\varphi}_k d\bar{\nu} \int \bar{\psi}_k d\bar{\nu}\right| &= \frac{1}{b_k}\left|\int (\bar{\varphi}_k \circ \bar{F}^n)\hat{\psi}_k d\bar{\nu} - \int \bar{\varphi}_k d\bar{\nu} \int \hat{\psi}_k d\bar{\nu}\right| \\ &\le \frac{1}{b_k}\int |\bar{\varphi}_k| \cdot \left|\frac{d(\bar{F}_*^n \hat{\lambda}_k)}{d\bar{m}} - \bar{\rho}\right| d\bar{m}. \qquad (16)\end{aligned}$$

Letting $\bar{\lambda}_k = \bar{F}_*^{2k}\hat{\lambda}_k$, we have

$$\frac{d}{d\bar{m}}\bar{F}_*^n \hat{\lambda}_k = \frac{d}{d\bar{m}}\bar{F}_*^{n-2k}\bar{\lambda}_k,$$

which together with (16) gives

$$\mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k; \bar{\nu}) \le \frac{1}{b_k}\|\bar{\varphi}_k\|_\infty \left|\bar{F}_*^{n-2k}\bar{\lambda}_k - \bar{\nu}\right| \le 3\|\psi\|_\infty \|\varphi\|_\infty \left|\bar{F}_*^{n-2k}\bar{\lambda}_k - \bar{\nu}\right|.$$

Let ϕ_k represent the density of the measure $\bar\lambda_k$ with respect to $\bar m$. The next lemma shows that $\phi_k \in \mathcal{F}_\beta^+$, with the constant C_{ϕ_k} not depending on ϕ_k. This is enough for using Theorem 3.6 and conclude the proof of Theorem B; recall that we have taken $k \le n/4$.

Lemma 4.1 *There is $C > 0$, not depending on ϕ_k, such that*

$$|\phi_k(\bar x) - \phi_k(\bar y)| \le C\beta^{\bar s(\bar x,\bar y)}, \quad \textit{for all } \bar x, \bar y \in \bar\Delta.$$

Proof Since $\bar F_*^{2k}\bar\nu = \bar\nu$ and $\bar\rho = d\bar\nu/d\bar m$, we may write

$$\bar\rho(\bar x) = \sum_{Q\in\bar{\mathcal{Q}}_{2k}} \frac{\bar\rho((\bar F^{2k}|Q)^{-1}(\bar x))}{J\bar F^{2k}((\bar F^{2k}|Q)^{-1}(\bar x))}. \tag{17}$$

Recall that we have by definition

$$\phi_k = \frac{d\bar\lambda_k}{dm} = \frac{d}{d\bar m}\bar F_*^{2k}\hat\lambda_k \quad \text{and} \quad \frac{d\hat\lambda_k}{dm} = \hat\psi_k\bar\rho.$$

Since $\hat\psi_k$ is constant on elements of $\mathcal{Q}_{2k}$, we have

$$\phi_k(\bar x) = \sum_{Q\in\bar{\mathcal{Q}}_{2k}} c_Q \frac{\bar\rho((\bar F^{2k}|Q)^{-1}(\bar x))}{J\bar F^{2k}((\bar F^{2k}|Q)^{-1}(\bar x))},$$

where c_Q is constant on each $Q \in \bar{\mathcal{Q}}_{2k}$. Hence,

$$\phi_k(\bar x) - \phi_k(\bar y) = \sum_{Q\in\bar{\mathcal{Q}}_{2k}} c_Q\left(\frac{\bar\rho((\bar F^{2k}|Q)^{-1}(\bar x))}{J\bar F^{2k}((\bar F^{2k}|Q)^{-1}(\bar x))} - \frac{\bar\rho((\bar F^{2k}|Q)^{-1}(\bar y))}{J\bar F^{2k}((\bar F^{2k}|Q)^{-1}(\bar y))}\right). \tag{18}$$

Fixing $Q \in \bar{\mathcal{Q}}_{2k}$, let $\bar x', \bar y', \in Q$ be such $\bar F^{2k}(\bar x') = \bar x$ and $\bar F^{2k}(\bar x') = \bar x$. We have

$$\frac{\bar\rho(\bar x')}{J\bar F^{2k}(\bar x')} - \frac{\bar\rho(\bar y')}{J\bar F^{2k}(\bar y')} = \left(\frac{\bar\rho(\bar y')}{J\bar F^{2k}(\bar y')}\right)\left(\frac{\bar\rho(\bar x')}{\bar\rho(\bar y')}\frac{J\bar F^{2k}(\bar y')}{J\bar F^{2k}(\bar x')} - 1\right). \tag{19}$$

It follows from Theorem 3.5 that there is $C_{\bar\rho} > 0$ such that

$$\log\left|\frac{\bar\rho(\bar x')}{\bar\rho(\bar y')}\right| \le C_{\bar\rho}\beta^{s(\bar x',\bar y')}.$$

On the other hand, by Lemma 3.4 there is $C_{\bar F} > 0$ such that

$$\log\left|\frac{J\bar F^{2k}(\bar y')}{J\bar F^{2k}(\bar x')}\right| \le C_{\bar F}\beta^{\bar s(\bar F^{2k}(\bar x'), F^{2k}(\bar y'))}.$$

Since $s(\bar x', \bar y') \ge s(\bar F^{2k}(\bar x'), F^{2k}(\bar y')) = s(\bar x, \bar y)$, we have

$$\log\left|\frac{\bar\rho(\bar x')}{\bar\rho(\bar y')}\frac{J\bar F^{2k}(\bar y')}{J\bar F^{2k}(\bar x')}\right| \le (C_{\bar\rho} + C_{\bar F})\beta^{\bar s(\bar x,\bar y)}. \tag{20}$$

Recalling (17) and the fact that $|c_Q| \le \|\hat{\psi}_k\|_\infty \le 3$, it follows from (18), (19) and (20) that there is some constant $C > 0$ not depending on ϕ_k such that

$$|\phi_k(\bar{x}) - \phi_k(\bar{y})| \le C\beta^{\bar{s}(\bar{x},\bar{y})}.$$

Actually, we may take $C = 3\|\bar{\rho}\|_\infty (C_{\bar{\rho}} + C_{\bar{F}})$. □

4.2 Central Limit Theorem

Let $\varphi \in H_\eta$ and consider its lift to Δ defined as $\tilde{\varphi} = \varphi \circ \pi$. Similarly to what we have done at the beginning of Sect. 4.1, we easily see that for proving Theorem C it is enough to obtain the Central Limit Theorem for $\tilde{\varphi}$ with respect to ν on Δ. As in the study of the correlations decay, the proof uses results from the quotient dynamics $\bar{F}\colon \bar{\Delta} \to \bar{\Delta}$. Let $\bar{\mathcal{B}}$ be the Borel σ-algebra on $\bar{\Delta}$. Define

$$\mathcal{B}_0 = \{\bar{\pi}^{-1}\bar{A}\colon \bar{A} \in \bar{\mathcal{B}}\} \quad \text{and} \quad \bar{\varphi}_0 = E_\nu(\tilde{\varphi} \mid \mathcal{B}_0).$$

Putting together the information from [10, Sect. 5.1.B] and Claim 1 in [10, Sect. 5.2] we easily see that it is enough to show that

$$\sum_{j\ge 0} \int |P^j(\bar{\varphi}_0\bar{\rho})|d\bar{m} < \infty,$$

where P is the transfer operator associated to $(\bar{F}, \bar{\nu})$. The proof of the Sublemma in [10, Sect. 5.2] gives that $\bar{\varphi}_0 \in \mathcal{F}_\beta$. Thus, if we consider $\bar{\lambda}$ the measure whose density with respect to $\bar{m}$ is $\bar{\varphi}_0\bar{\rho}$, then $P^j(\bar{\varphi}_0\bar{\rho})$ is by definition the density of $F_*^j\bar{\lambda}$ with respect to $\bar{m}$. Hence, we just have to show that

$$\sum_{j\ge 0} \int \left|\frac{d}{d\bar{m}}\bar{F}_*^j\bar{\lambda}\right| d\bar{m} < \infty.$$

First we "renormalize" $\bar{\lambda}$. Let

$$b = \left(\int (\bar{\varphi}_0 + \|\bar{\varphi}_0\|_\infty) d\bar{\nu}\right)^{-1} \quad \text{and} \quad \hat{\varphi}_0 = b(\bar{\varphi}_0 + 2\|\bar{\varphi}_0\|_\infty),$$

and consider $\hat{\lambda}$ the probability measure whose density with respect to $\bar{m}$ is $\hat{\varphi}_0\bar{\rho}$. We have

$$\int \hat{\varphi}_0 d\bar{\nu} = \int \hat{\varphi}_0\bar{\rho}d\bar{m} = 1. \tag{21}$$

Recalling that

$$\int \bar{\varphi}_0 d\bar{\nu} = \int \bar{\varphi}_0\bar{\rho}d\bar{m} = 0,$$

we may write

$$\int \left|\frac{d}{d\bar{m}}\bar{F}_*^j\bar{\lambda}\right| d\bar{m} = \int \left|\frac{d}{d\bar{m}}\bar{F}_*^j\bar{\lambda} - \bar{\rho}\int \bar{\varphi}_0 d\bar{\nu}\right| d\bar{m}.$$

Using (21) and the fact that

$$\frac{d}{d\bar{m}}\bar{F}_*\bar{\nu} = \frac{d}{d\bar{m}}\bar{\nu} = \bar{\rho},$$

we obtain

$$\begin{aligned}\int\left|\frac{d}{d\bar{m}}\bar{F}_*^j\bar{\lambda}\right|d\bar{m} &= \frac{1}{b}\int\left|\frac{d}{d\bar{m}}\bar{F}_*^j\hat{\lambda} - 2\bar{\rho}\|\bar{\varphi}_0\|_\infty - \bar{\rho}\int\hat{\varphi}_0 d\bar{\nu} + 2\bar{\rho}\|\bar{\varphi}_0\|_\infty\right|d\bar{m}\\ &= \frac{1}{b}\int\left|\frac{d}{d\bar{m}}\bar{F}_*^j\hat{\lambda} - \bar{\rho}\right|d\bar{m}\\ &= \frac{1}{b}|\bar{F}_*^j\hat{\lambda} - \bar{\nu}|.\end{aligned}$$

Under the hypotheses of Theorem C this last quantity is clearly summable, by Theorem 3.6.

5 A Solenoid with Intermittency

Here we construct a hyperbolic structure for the map g defined in Sect. 2.4 which satisfies the assumptions of our main theorems. Concerning **(P$_1$)**–**(P$_4$)**, we just have to show that **(P$_1$)** and **(P$_4$)** hold, since **(P$_2$)** and **(P$_3$)** are trivially satisfied due to the uniform contraction of g in the vertical direction and the skew-product form of g. We also need to give suitable estimates for the decay of return times and the diameters in (2). The conclusions of Theorem D are then a consequence of our main results.

The map g possesses an attractor in M which is precisely

$$\Sigma = \bigcap_{n\geq 0} g^n(M).$$

Σ is locally a product of an interval by a Cantor set. Topologically this set coincides with the solenoid attractor for the classical case where f is taken uniformly expanding in S^1.

For defining the hyperbolic structure we are going to construct a (mod 0) countable partition $\mathcal{P}_0$ of an interval $I_1 \subset S^1$ and associate to each element of $\mathcal{P}_0$ a suitable return time R^* with respect to the map f. Then we take

$$\Lambda = \Sigma \cap (I_1 \times D^2).$$

For each $(x, y) \in \Lambda$ we define $\gamma^s(x, y) = \{(x, y)\colon y \in D^2\}$ and $\gamma^u(x, y)$ as the connected component of Λ that contains (x, y). The s-subsets are precisely the sets $\Sigma \cap (P \times D^2)$ with $P \in \mathcal{P}_0$ and the return times are taken accordingly.

5.1 Partition and Return Times

Here we recall some objects and results from [11, Sect. 6] related to the map f. Let $I_1, \ldots, I_d$ be the partition of S^1 made by the fundamental domains of f arranged in a natural order, and assume for definiteness that 0 is the common endpoint of I_1 and I_d. Letting x_0 be the other endpoint of I_1 we define a sequence $(x_n)_n$ in I_1 with the property that $f(x_{n+1}) = x_n$ for $n \geq 0$. Likewise, we consider x'_0 the endpoint of I_d distinct from 0 and define a sequence $(x'_n)_n$ in I_d so that $f(x'_{n+1}) = x'_n$ for $n \geq 0$.

Let $J_n = [x_{n+1}, x_n]$ and $J'_n = [x'_n, x'_{n+1}]$ for $n \geq 0$. Consider the (mod 0) partition of S^1

$$\mathcal{A} = \{I_2, \ldots, I_{d-1}\} \cup \{J_n, J'_n;\ n \geq 0\}.$$

Let $R = 1$ on $I_2 \cup \cdots \cup I_{d-1} \cup J_0 \cup J'_0$ and let $R|J_n = R|J'_n = n+1$ for $n \geq 1$. We have $f^R(I_j) = S^1$ for $2 \leq j \leq d-1$ and the f^R images of all other elements of $\mathcal{A}$ are either $I_1 \cup \cdots \cup I_{d-1}$ or $I_2 \cup \cdots \cup I_d$. The following results were proved in [11, Sects. 6.2 and 6.3]:

(1) Tail decay: $\mathrm{Leb}\{R > n\} \approx n^{-1/\gamma}$;
(2) Expansion: *there is* $0 < \beta < 1$ *such that* $(f^R)'(x) \geq \beta^{-1}$ *for every* $x \in S^1 \setminus \{0\}$;
(3) Bounded distortion: *there is* $C > 0$ *such that for every* $1 \leq i \leq n$ *and* $x, y \in J_n$

$$\log \frac{(f^i)'(x)}{(f^i)'(y)} \leq C \frac{|f^i(x) - f^i(y)|}{|J_{n-i}|}.$$

This function R does not qualify as a return time for a hyperbolic structure of f satisfying the Markov property. That role will be played by the function R^* we introduce below. We use the time function R to define a sequence of stopping times $(S_i)_i$ as in (1). We also define the sequence of return times

$$r_1 = S_1 = R, \quad \text{and} \quad r_{i+1} = S_{i+1} - S_i, \quad \text{for } i \geq 1.$$

Using this sequence of stopping times we define the first return time R^* to I_1 as follows. We simply take $R^*(x) = S_i(x)$, where $i \geq 1$ is the minimum such that $f^{S_i}(x) \in I_1$. As shown in [11, Sect. 6.2] we have

$$\mathrm{Leb}\{R^* > n\} \lesssim n^{-1/\gamma}. \tag{22}$$

Let $\mathcal{P}_0$ be the Markov partition of I_1 associated to R^*. Naturally associating the return times to the s-subsets described above, then (22) gives the tail estimate that we need. Property **(P_4)** is an easy consequence of the bounded distortion and expansion above, since the estimates on the derivative of g in the unstable direction are given by f.

5.2 Diameter Estimate

Now we are going to show that $\delta_k \lesssim 1/k^{1/\gamma}$, where δ_k is the quantity defined in (2). Taking into account the uniform contraction on the stable direction, we just have to obtain the desired control on the unstable one. We start by proving the following auxiliary result.

Lemma 5.1 *Let X be an interval in S^1 whose points have the same stopping times $S_1, \ldots, S_N$, for some $N \geq 1$, with $S_N \geq k$, and such that $f^{S_N}(X) = I_1$. Then $|X| \lesssim 1/k^{1/\gamma}$.*

Proof Let $r_1, \ldots, r_N$ be the return times of points in X. Since we are assuming that $S_N = r_1 + \cdots + r_N \geq k$, there must be some $1 \leq m \leq N$ such that $r_m \geq k/N$. Let

$$Y = f^{r_1 + \cdots + r_{m-1}}(X).$$

Considering the interval $I \in \mathcal{A}$ such that $Y \subset I$ we have $R|I = r_m$. Bounded distortion yields

$$|Y| \lesssim \frac{|f^{r_m}(Y)|}{|f^{r_m}(I)|} \cdot |I|.$$

On the other hand, the tail decay and expansion estimates give

$$|I| \lesssim \left(\frac{1}{r_m}\right)^{1/\gamma}, \qquad |X| \le \beta^{m-1}|Y| \quad \text{and} \quad |f^{r_m}(Y)| \le \beta^{N-m}|I_1|.$$

Taking into account the choice of m we obtain

$$|X| \lesssim \beta^N \left(\frac{N}{k}\right)^{1/\gamma} \lesssim \frac{1}{k^{1/\gamma}},$$

and so we are done. □

Take $P \in \mathcal{P}_0$ and $k \ge 1$. We consider the three possible cases of sets whose diameters have to be controlled. The first two correspond to $k \le R^*(P) - 1$, and the last one corresponds to $k > R^*(P) - 1$; recall the definition of δ_k in Sect. 2.2.

Case 1 Assume first that $k \le R^*(P) - 1$ and $0 \le \ell < R^*(P) - k$. There is $m \ge 1$ such that $R^*(P) = r_1 + \cdots + r_m$. Considering $r_0 = 0$, let $0 \le p < m$ be such that

$$r_0 + \cdots + r_p \le \ell < r_0 + \cdots + r_{p+1}.$$

Letting $\ell' = \ell - (r_0 + \cdots + r_p)$, we have

$$\ell' < r_{p+1} \quad \text{and} \quad \ell' < r_{p+1} + \cdots + r_m - k.$$

Now, if $i + 1 = m$, then

$$|f^{\ell}(P)| = |f^{\ell'}(f^{r_0+\cdots+r_p}(P))| \lesssim \left(\frac{1}{r_m - \ell'}\right)^{1/\gamma} \le \frac{1}{k^{1/\gamma}}.$$

Otherwise, for $p + 1 < m$ we use Lemma 5.1 with $X = f^{r_0+\cdots+r_p+\ell'}(P)$, $N = m - p$, and

$$S_j = r_{p+1} + \cdots + r_{p+j} - \ell', \quad \text{for } 1 \le j \le N,$$

thus obtaining

$$|f^{\ell}(P)| = |X| \lesssim \frac{1}{k^{1/\gamma}}.$$

Observe that $r_{p+1} - \ell'$ is still a return time, which then implies that S_1 is well defined.

Case 2. Assume now that $k \le R^*(P) - 1$ and $R^*(P) - k \le \ell \le R^*(P) - 1$. We simply write R^* for $R^*(P)$. Take $Q \in \mathcal{P}_{k-R^*+1+\ell}$ with $Q \subset P$. By construction, there is $j \ge 0$ such that points in $f^{R^*}(Q)$ have the same stopping times $S_1^*, \ldots, S_j^*$ up to time $k - R^* + \ell$. Moreover, $f^{S_{j+1}^*}(f^{R^*}(Q)) = I_1$ and

$$S_{j+1}^* \ge k - R^* + 1 + \ell. \tag{23}$$

There are integers $m, n \ge 1$ and return times $r_1, \ldots, r_{m+n}$ such that

$$R^* = r_1 + \cdots + r_m \quad \text{and} \quad S_{j+1}^* = r_{m+1} + \cdots + r_{m+n}. \tag{24}$$

It follows from (23) and (24) that

$$\ell < r_1 + \cdots + r_{m+n} - k.$$

Considering $0 \le p < m$ such that

$$r_0 + \cdots + r_p \le \ell < r_0 + \cdots + r_{p+1},$$

where $r_0 = 0$ as before, and taking $\ell' = \ell - (r_0 + \cdots + r_p)$, we have

$$\ell' < r_{p+1} \quad \text{and} \quad \ell' < r_{p+1} + \cdots + r_{m+n} - k.$$

The proof now follows as in the previous case.

Case 3. The case $k > R(P) - 1 \ge \ell$ is treated as Case 2.

Appendix A: Mixing Rates for Tower Maps

The goal of this section is to prove Theorem 3.6. We follow the scheme of [11] with a delicate control on the constants. The only exception is Appendix A.2.2 where we use results from [4]. The setting will be the same of Sect. 3.3. For the sake of notational simplicity we shall drop all bars.

Let λ and λ' be probability measures in Δ whose densities with respect to m belong to $\mathcal{F}_\beta^+$. Let

$$\varphi = \frac{d\lambda}{dm} \quad \text{and} \quad \varphi' = \frac{d\lambda'}{dm},$$

and consider C_φ, $C_{\varphi'}$ as in the definition of $\mathcal{F}_\beta^+$.

A.1 Main Estimates

Consider the product map $F \times F : \Delta \times \Delta \to \Delta \times \Delta$, and $P = \lambda \times \lambda'$ the product measure on $\Delta \times \Delta$. Let $\pi, \pi' : \Delta \times \Delta \to \Delta$ be the projections on the first and second coordinates respectively. Note that $F^n \circ \pi = \pi \circ (F \times F)^n$. Consider the partition $\mathcal{Q} := \{\Delta_{l,i}\}$ of Δ, and the partition $\mathcal{Q} \times \mathcal{Q}$ of $\Delta \times \Delta$. Note that each element of $\mathcal{Q} \times \mathcal{Q}$ is sent bijectively by $F \times F$ onto a union of elements of $\mathcal{Q} \times \mathcal{Q}$. For each $n \ge 1$, let

$$(\mathcal{Q} \times \mathcal{Q})_n := \bigvee_{i=0}^{n-1} (F \times F)^{-i} (\mathcal{Q} \times \mathcal{Q}),$$

and let $(\mathcal{Q} \times \mathcal{Q})_n(x, x')$ be the element of $(\mathcal{Q} \times \mathcal{Q})_n$ that contains $(x, x') \in \Delta \times \Delta$.

Since (F, ν) is mixing and the density of ν with respect to m belongs to $L^\infty(m)$, we may find $n_0 \in \mathbb{N}$ and $\gamma_0 > 0$ such that $m(F^{-n}(\Delta_0) \cap \Delta_0) \ge \gamma_0$ for all $n \ge n_0$. Then we introduce a sequence of *stopping times* $0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots$ in $\Delta \times \Delta$ given by

$$\begin{aligned}
\tau_1(x, x') &= n_0 + \hat{R}(F^{n_0}(x)), \\
\tau_2(x, x') &= \tau_1 + n_0 + \hat{R}(F^{\tau_1}(x')), \\
\tau_3(x, x') &= \tau_2 + n_0 + \hat{R}(F^{\tau_2}(x)), \\
\tau_4(x, x') &= \tau_3 + n_0 + \hat{R}(F^{\tau_3}(x')), \\
&\vdots
\end{aligned}$$

with the falls to the ground level Δ_0 alternating between x e x'. This implies that $\tau_{i+1} - \tau_i \geq n_0$ for all $i \geq 1$. We define the *simultaneous return time* $T : \Delta \times \Delta \to \mathbb{N}$ as

$$T(x, x') = \min\big\{\tau_i : (F^{\tau_i}(x), F^{\tau_i}(x')) \in \Delta_0 \times \Delta_0, \text{ with } i \geq 2\big\}.$$

Note that we have $T \geq 2n_0$. Since (F, ν) is mixing, then $(F \times F, \nu \times \nu)$ is ergodic, and so T is well-defined $m \times m$ almost everywhere. Observe that if $T(x, x') = n$, then

$$T|_{(\mathcal{Q}\times\mathcal{Q})_n(x,x')} \equiv n \quad \text{and} \quad (F \times F)^n((\mathcal{Q} \times \mathcal{Q})_n(x, x')) = \Delta_0 \times \Delta_0.$$

Now we define a sequence $\xi_1 < \xi_2 < \xi_3 < \cdots$ of partitions of $\Delta \times \Delta$. First we take $\xi_1(x, x') = (F^{-\tau_1(x)+1}\mathcal{Q})(x) \times \Delta$. The partition ξ_1 is formed by sets of the form $\Gamma = A \times \Delta$ where τ_1 is constant on Γ and F^{τ_1} sends A bijectively to Δ_0. For $i > 1$, if i if even (resp. odd), we define ξ_i as the refinement of ξ_{i-1} obtained by partitioning $\Gamma \in \xi_{i-1}$ in the x' direction (resp. x direction) into sets $\tilde{\Gamma}$ such that τ_i is constant on each $\tilde{\Gamma}$ and F^{τ_i} sends $\pi'(\tilde{\Gamma})$ (resp. $\pi(\tilde{\Gamma})$) bijectively to Δ_0. It will be useful to consider $\xi_0 = \{\Delta \times \Delta\}$. Let us mention two useful properties about the measurability of the functions with respect to the partitions defined above:

- $\tau_1, \tau_2, \ldots, \tau_i$ are ξ_i-measurable for each $i \geq 1$;
- $\{T = \tau_i\}$ and $\{T > \tau_i\}$ are ξ_{i+1}-measurable for each $i \geq 1$.

This follows from the construction of the objects. Now we present the main estimates we need on $\{\tau_i\}$ and T, whose proofs we postpone to Appendix A.3.

(E_1) There is $\varepsilon_0 = \varepsilon_0(C_\varphi, C_{\varphi'}) > 0$ such that $P\{T = \tau_i \mid \Gamma\} \geq \varepsilon_0$ for $i \geq 2$ and $\Gamma \in \xi_i$ with $T \mid \Gamma > \tau_{i-1}$. The dependence of ε_0 on C_φ and $C_{\varphi'}$ can be removed if we consider $i \geq i_0(C_\varphi, C_{\varphi'})$.

(E_2) There is $K_0 = K_0(C_\varphi, C_{\varphi'}) > 0$ such that $P\{\tau_{i+1} - \tau_i > n_0 + n \mid \Gamma\} \leq K_0 m\{\hat{R} > n\}$ for $i \geq 0$, $\Gamma \in \xi_i$ and $n \geq 0$. The dependence of K_0 on C_φ and $C_{\varphi'}$ can be removed if we consider $i \geq i_0(C_\varphi, C_{\varphi'})$.

Let $0 \equiv T_0 < T_1 < T_2 < \cdots$ be *stopping times* in $\Delta \times \Delta$ given by

$$T_1 = T, \quad \text{and} \quad T_n = T_{n-1} + T \circ (F \times F)^{T_{n-1}}, \quad \text{for } n \geq 2. \tag{25}$$

(E_3) There are $K_1 = K_1(C_\varphi, C_{\varphi'}) > 0$ and $\varepsilon_1 > 0$ (not depending on φ or φ') such that $\left|F_*^n\lambda - F_*^n\lambda'\right| \leq 2P\{T > n\} + K_1 \sum_{i=1}^{\infty}(1 - \varepsilon_1)^i P\{T_i \leq n < T_{i+1}\}$ for $n \geq 1$.

(E_4) There is $K_2 = K_2(C_\varphi, C_{\varphi'}) > 0$ such that $P\{T_{i+1} - T_i > n\} \leq K_2(m \times m)\{T > n\}$ for $i \geq 0$.

A.2 Convergence to the Equilibrium

We shall use (E_1)–(E_4) to prove Theorem 3.6. Let ν be the measure given by Theorem 3.5. Observe that ν is a fixed point for F_*, whose density with respect to m belongs to $\mathcal{F}_\beta^+$. Theorem 3.6 follows just by taking $\lambda' = \nu$, once we obtain the upper bound for $\left|F_*^n\lambda - F_*^n\lambda'\right|$.

We start by observing that for each $i \geq 1$ we have

$$P\{T_i \leq n < T_{i+1}\} \leq \sum_{j=0}^{i} P\left\{T_{j+1} - T_j > \frac{n}{i+1}\right\}. \tag{26}$$

Actually, since we have $T_{i+1} > n$, there must be some $0 \le j \le i$ with $T_{j+1} - T_j > n/(i+1)$. For otherwise

$$T_{i+1} = \sum_{j=0}^{i} (T_{j+1} - T_j) \le \sum_{j=0}^{i} \frac{n}{i+1} = n,$$

which is an absurd. Hence

$$\{T_i \le n < T_{i+1}\} \subset \bigcup_{j=0}^{i} \left\{ T_{j+1} - T_j > \frac{n}{i+1} \right\},$$

which gives (26). It follows respectively from (E_3), (26) and (E_4) that

$$\begin{aligned}
\left| F_*^n \lambda - F_*^n \lambda' \right| &\le 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} (1-\varepsilon_1)^i P\{T_i \le n < T_{i+1}\}, \\
&\le 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} (1-\varepsilon_1)^i \sum_{j=0}^{i} P\left\{ T_{j+1} - T_j > \frac{n}{i+1} \right\}, \\
&\le 2P\{T > n\} + K_1 K_2 \sum_{i=1}^{\infty} (1-\varepsilon_1)^i (i+1)(m \times m) \left\{ T > \frac{n}{i+1} \right\}.
\end{aligned}$$

Observe that both in the polynomial and stretched exponential cases, as long we obtain the desired decay for $P\{T > n\}$, then taking $P = m \times m$ it immediately follows that

$$\sum_{i=1}^{\infty} (1-\varepsilon_1)^i (i+1)(m \times m) \left\{ T > \frac{n}{i+1} \right\}$$

decays at the same speed of $P\{T > n\}$. Consequently, we are left to estimate $P\{T > n\}$. At this point we distinguish the polynomial and stretched exponential cases.

A.2.1 Polynomial Decay

Assume there are $C > 0$ and $\alpha > 1$ such that $m\{R > n\} \le Cn^{-\alpha}$ for all $n \ge 1$. Then, there is $\hat{C} > 0$ (depending only on C and α) such that

$$m\{\hat{R} > n\} = \sum_{l > n} m\{R > l\} \le \hat{C} n^{-\alpha+1}. \tag{27}$$

Recall that $T \ge 2n_0$ by construction. We write

$$P\{T > n\} = \sum_{1 \le i < [\frac{n}{2n_0}]} P\{T > n \colon \tau_i \le n < \tau_{i+1}\} + P\{T > n \colon \tau_{[\frac{n}{2n_0}]} \le n\}. \tag{28}$$

Since $\{T > \tau_{i-1}\}$ is ξ_i-measurable, conditioning on the elements of the partition ξ_i and using (E_1) it yields for $i \ge 2$

$$P\{T > \tau_i \mid T > \tau_{i-1}\} = 1 - P\{T = \tau_i \mid T > \tau_{i-1}\} \ge 1 - \varepsilon_0. \tag{29}$$

From (29) we obtain for $n \geq 4n_0$

$$P\{T > n \colon \tau_{[\frac{n}{2n_0}]} \leq n\} \leq P\{T > \tau_{[\frac{n}{2n_0}]}\}$$
$$= P\{T > \tau_1\} \cdot \prod_{i=2}^{[\frac{n}{2n_0}]} P\{T > \tau_i \mid T > \tau_{i-1}\}$$
$$\leq (1-\varepsilon_0)^{[\frac{n}{2n_0}]-1}. \tag{30}$$

Since the dependence of ε_0 on P can be removed if we consider $i \geq i_0$ for some $i_0 = i_0(P)$, we are left to compute the decay of

$$\sum_{1 \leq i < [\frac{n}{2n_0}]} P\{T > n \colon \tau_i \leq n < \tau_{i+1}\}. \tag{31}$$

For each $i \geq 1$ we have $P\{T > n \colon \tau_i \leq n < \tau_{i+1}\} \leq P\{T > \tau_i : n < \tau_{i+1}\}$. As in (26) we may show that

$$\{T > \tau_i : n < \tau_{i+1}\} \subset \bigcup_{j=0}^{i} \left\{T > \tau_i : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\},$$

which then gives

$$P\{T > n \colon \tau_i \leq n < \tau_{i+1}\} \leq \sum_{j=0}^{i} P\left\{T > \tau_i : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\}. \tag{32}$$

Our next goal is to estimate the terms in the sum (32). Consider first the terms with $i, j \geq 2$. We write

$$P\left\{T > \tau_i : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} = A \cdot B \cdot C, \tag{33}$$

with

$$A = P\{T > \tau_1\} \cdot \prod_{k=2}^{j-1} P\{T > \tau_k \mid T > \tau_{k-1}\}$$
$$B = P\left\{T > \tau_j : \tau_{j+1} - \tau_j > \frac{n}{i+1} \mid T > \tau_{j-1}\right\};$$
$$C = \prod_{k=j+1}^{i} P\left\{T > \tau_k \mid T > \tau_{k-1} : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\}.$$

Observe that $A = P\{T > \tau_1\}$ when $j = 2$, and C is void when $j = i$. Arguing as in (29), from estimate (E_1) one gets

$$A \leq (1-\varepsilon_0)^{j-2}. \tag{34}$$

Conditioning on ξ_k and using (E_1), we have that each term in C is also bounded from above by $1-\varepsilon_0$, which then gives

$$C \leq (1-\varepsilon_0)^{i-j}. \tag{35}$$

Since $\{T > \tau_{i-1}\}$ is ξ_i-measurable, conditioning on elements of ξ_i and using (E_2) we get

$$B \le P\left\{\tau_{j+1} - \tau_j > \frac{n}{i+1} \mid T > \tau_{j-1}\right\} \le K_0 m\left\{\hat{R} > \frac{n}{i+1} - n_0\right\}. \tag{36}$$

Using (27) and the fact that $i < [\frac{n}{2n_0}]$ we obtain

$$B \le K_0\hat{C}\left(\frac{n}{i+1} - n_0\right)^{-\alpha+1} \le K_0\hat{C}2^{1-\alpha}\left(\frac{n}{i+1}\right)^{-\alpha+1}.$$

From (33)–(36) we deduce for $i, j \ge 2$

$$P\left\{T > \tau_i : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \le K_0\hat{C}2^{1-\alpha}\left(\frac{n}{i+1}\right)^{-\alpha+1}(1-\varepsilon_0)^{i-2}. \tag{37}$$

Let us consider now the small terms in the sum (32). For $i \ge 2$ and $j = 0, 1$ we write

$$\begin{aligned} P&\left\{T > \tau_i : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \\ &\le P\left\{T > \tau_1 : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \\ &\quad \times \prod_{k=2}^{i} P\left\{T > \tau_k \mid T > \tau_{k-1} : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \\ &\le P\left\{\tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \cdot \prod_{k=2}^{i} P\left\{T > \tau_k \mid T > \tau_{k-1} : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\}. \end{aligned}$$

We treat this case arguing as before, thus obtaining

$$P\left\{T > \tau_i : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \le K_0\hat{C}2^{1-\alpha}\left(\frac{n}{i+1}\right)^{-\alpha+1}(1-\varepsilon_0)^{i-1}. \tag{38}$$

Finally, for $i = 1$ and $j = 0, 1$, we have

$$P\left\{T > \tau_1 : \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \le P\left\{\tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} \le K_0\hat{C}2^{1-\alpha}\left(\frac{n}{i+1}\right)^{-\alpha+1}. \tag{39}$$

Using (32) and (37)–(39) we get $P\{T > n : \tau_i \le n < \tau_{i+1}\} \le C_0(1-\varepsilon_0)^i(i+1)^\alpha n^{-\alpha+1}$, where C_0 is a constant depending only on K_0, $\hat{C}$, α and ε_0. This yields the desired bound for (31) in the polynomial case.

A.2.2 Stretched Exponential Decay

Assume that there are $C, c > 0$ and $0 < \eta \le 1$ such that $\mathrm{Leb}\{R > n\} \le Ce^{-cn^\eta}$ for all $n \ge 1$. Then there is $\hat{C} > 0$ such that

$$m\{\hat{R} > n\} = \sum_{l>n} m\{R > l\} \le \hat{C}e^{-cn^\eta}.$$

The conclusion in this case is a consequence of (E_1)–(E_2) and the next lemma, which can easily be obtained from [4, Lemma 4.2] by taking $L = 1$, $\tau = T$, $\mu = P$ and $t_j = \tau_j$.

Lemma A.1 *Assume that there are* $\varepsilon_0 > 0$ *and* $K_0 > 0$ *such that for all* $i \geq 2$ *and* $\Gamma \in \xi_i$ *with* $T \mid \Gamma > \tau_{i-1}$ *we have*

(1) $P\{T = \tau_i \mid \Gamma\} \geq \varepsilon_0$;
(2) $P\{\tau_{i+1} - \tau_i > n \mid \Gamma\} \leq K_0 e^{-cn^\eta}$.

Then there exist $C', c' > 0$ *such that* $P\{T > n\} \leq C' e^{-c' n^\eta}$.

For the sake of completeness one must verify that the constants C' and c' obey the final requirement of Theorem 3.6. Actually, it is proved in [4, Lemma 4.2] that there are a measurable function k and a measurable set B_n such that for $q(n) = [\alpha n^\eta]$, with small $\alpha > 0$, we have $\{T > n\} \subset \{k > q(n)\} \cup B_n$ (recall estimate (27) in [4]) with $P\{k > q(n)\} \leq (1 - \varepsilon_0)^{q(n)}$, and for some positive integer K only depending on K_0 and η,

$$P(B_n) \leq 2^{q(n)} \sum_{p \geq n/2} C e^{-cp^\eta}, \quad \text{for } n \geq K.$$

Hence, taking $\alpha > 0$ sufficiently small we obtain the desired conclusion.

A.3 Main Estimates

Here we obtain estimates (E_1)–(E_4). We start with some preliminary results on distortion control that will enable us to prove (E_1) and (E_2).

Lemma A.2 *There is* $C_0 = C_0(C_\varphi) > 0$ *such that for every* $k \geq 1$ *and* $A \in \vee_{i=0}^{k-1} F^{-i} \mathcal{Q}$ *with* $F^k(A) = \Delta_0$, *we have for* $\mu = F_*^k(\lambda|A)$ *and all* $x, y \in \Delta_0$

$$\left| \frac{d\mu}{dm}(x) \Big/ \frac{d\mu}{dm}(y) \right| \leq C_0.$$

Moreover, the dependence of C_0 *on* C_φ *can be removed if we assume that the number of visits* $j \leq k$ *of* A *to* Δ_0 *is bigger than some* $j_0 = j_0(C_\varphi)$.

Proof Let $x_0, y_0 \in A$ be such that $F^k(x_0) = x$ and $F^k(y_0) = y$. Using (12) and the fact that $\varphi \in \mathcal{F}_\beta$ we have

$$\left| \frac{d\mu}{dm}(x) \Big/ \frac{d\mu}{dm}(y) \right| = \left| \frac{\varphi(x_0)}{JF^k(x_0)} \cdot \frac{JF^k(y_0)}{\varphi(y_0)} \right| \leq \frac{\varphi(x_0)}{\varphi(y_0)} \cdot \left| \frac{JF^k(y_0)}{JF^k(x_0)} \right| \leq (1 + C_\varphi \beta^j)(1 + C_F),$$

where j is the number of visits of A to Δ_0 prior to k. □

The next result is proved in [11, Sublemma 1].

Lemma A.3 *There is* $M_0 > 0$ *such that* $\frac{dF_*^n m}{dm} \leq M_0$ *for all* $n \geq 1$.

A.3.1 Proof of (E_1)

Assume without loss of generality that i is even, and take $\Gamma \in \xi_i$ as in the statement of (E_1). We have $\Gamma = A \times B$ with $A, B \subset \Delta$, where A is sent bijectively by $F^{\tau_{i-1}}$ to Δ_0 and $F^{\tau_{i-1}}(B)$ is contained in some $\Delta_{l,j}$. At time τ_i we have $F^{\tau_i}(B) = \Delta_0$ and $F^{\tau_i}(A)$ is spread over several parts of $\bigcup\{\Delta_l : l \leq \tau_i - \tau_{i-1}\}$. The set $\{T = \tau_i\} \cap \Gamma$ has the form $A' \times B$ where A' is the set of points in A which are sent to Δ_0 by $F^{\tau_{i-1}}$ and return to Δ_0 by $F^{\tau_i-\tau_{i-1}}$. Letting $\mu = F_*^{\tau_{i-1}}(\lambda|A)$ we may write

$$P\{T=\tau_i \mid \Gamma\} = \frac{\lambda(A')}{\lambda(A)} = \frac{\mu(F^{-(\tau_i-\tau_{i-1})}(\Delta_0)\cap\Delta_0)}{\mu(\Delta_0)}.$$

Note that Lemma A.2 applies to μ, thus giving

$$P\{T=\tau_i \mid \Gamma\} \geq C_0^{-2}\frac{m(F^{-(\tau_i-\tau_{i-1})}(\Delta_0)\cap\Delta_0)}{m(\Delta_0)}.$$

Recall that n_0 has been chosen in such a way that there is γ_0 such that $m(F^{-n}(\Delta_0)\cap\Delta_0)\geq \gamma_0>0$, for all $n\geq n_0$. By construction we have $\tau_i-\tau_{i-1}\geq n_0$. This is enough for concluding that there is some $\varepsilon_0=\varepsilon_0(C_\varphi)>0$ for which $P\{T=\tau_i \mid \Gamma\}\geq\varepsilon_0$. The other case ($i$ odd) gives the dependence of ε_0 also on $C_{\varphi'}$. These dependencies can be removed if we take i large enough, according to Lemma A.2.

A.3.2 Proof of (E_2)

For $i=0$ we have

$$P\{\tau_1>n_0+n\} = (F_*^{n_0}\lambda)\{\hat R>n\} \leq \left\|\frac{d\lambda}{dm}\right\|_\infty M_0 m\{\hat R>n\},$$

and for $i=1$

$$P\{\tau_2-\tau_1>n_0+n\} = \left(F_*^{\tau_1+n_0}\lambda'\right)\{\hat R>n\} \leq \left\|\frac{d\lambda'}{dm}\right\|_\infty M_0 m\{\hat R>n\},$$

which obviously give upper bounds depending on C_φ and $C_{\varphi'}$.

Let us consider now the case $i\geq 2$. Assume for definiteness that i is even. Considering the probability measure

$$\mu = \frac{1}{P(\Gamma)}F_*^{\tau_{i-1}}\pi_*(P|\Gamma)$$

we have

$$\begin{aligned} P\{\tau_{i+1}-\tau_i>n_0+n \mid \Gamma\} &= \left(F_*^{(\tau_i-\tau_{i-1})+n_0}\mu\right)\{\hat R>n\} \\ &\leq \left\|\frac{d}{dm}\left(F_*^{(\tau_i-\tau_{i-1})+n_0}\mu\right)\right\|_\infty m\{\hat R>n\} \\ &\leq M_0\left\|\frac{d\mu}{dm}\right\|_\infty m\{\hat R>n\}, \quad \text{by Lemma A.3.} \end{aligned}$$

Using Lemma A.2 one has that $\|d\mu/dm\|_\infty$ is bounded from above by some constant only depending on C_0. Moreover, according to Lemma A.2, this dependency can be removed if we take i large enough.

For obtaining (E_3) and (E_4) we consider the dynamical system $\hat F=(F\times F)^T:\Delta\times\Delta\circlearrowleft$. It follows from the definition of the sequence $\{T_n\}$ in (25) that

$$\hat F^n = (F\times F)^{T_n}, \quad \text{for all } n\geq 1. \tag{40}$$

Let $\hat{\xi}_1$ denote the partition into rectangles $\hat{\Gamma}$ of $\Delta \times \Delta$ on which T is constant and $\hat{F}^n$ maps $\hat{\Gamma}$ bijectively to $\Delta_0 \times \Delta_0$. Next we define inductively partitions $\hat{\xi}_2, \hat{\xi}_3, \dots$ of $\Delta \times \Delta$ by $\hat{\xi}_n := \hat{F}^{-(n-1)}\hat{\xi}_1$, for $n \geq 2$. Each $\hat{\xi}_n$ is the partition into subsets $\hat{\Gamma}$ of $\Delta \times \Delta$ on which T_n is constant and $\hat{F}$ maps $\hat{\Gamma}$ bijectively to $\Delta_0 \times \Delta_0$. We consider the reference measure $m \times m$ for the dynamical system $\hat{F}$ and $J\hat{F}$ the Jacobian of $\hat{F}$ with respect to $m \times m$. We define a separation time $\hat{s} : (\Delta \times \Delta) \times (\Delta \times \Delta) \to \mathbb{N}_0$ for $\hat{F}$ in the following way: given $w, z \in \Delta \times \Delta$, take

$$\hat{s}(w, z) = \min\left\{n \geq 0\colon\ \hat{F}^n w \text{ and } \hat{F}^n z \text{ lie in distinct elements of } \hat{\xi}_1\right\}.$$

Denoting

$$\Phi = \frac{dP}{d(m \times m)},$$

we have $\Phi(x, x') = \varphi(x)\varphi'(x')$. With no loss of generality we assume from here on that $\varphi(x) > 0$ and $\varphi(y) > 0$. The next two results are proved in [11, Sublemma 3].

Lemma A.4 *Let $C_1 > 0$ be as in Lemma 3.4. Given $w, z \in \Delta \times \Delta$ with $\hat{s}(w, z) \geq n \geq 1$*

$$\log \frac{J\hat{F}^n(w)}{J\hat{F}^n(z)} \leq 2C_1\beta^{\hat{s}(\hat{F}^n(w), \hat{F}^n(z))}.$$

Lemma A.5 *Let $C_\Phi = C_\varphi + C_{\varphi'}$. Given $w, z \in \Delta \times \Delta$*

$$\log \frac{\Phi(w)}{\Phi(z)} \leq C_\Phi \beta^{\hat{s}(w,z)}.$$

The next result gives a distortion control similar to that of Lemma A.2.

Lemma A.6 *There is $C_* = C_*(C_\varphi, C_{\varphi'}) > 0$ such that for any $i \geq 1$ and any $\Gamma \in \hat{\xi}_i$, we have for all $x, y \in \Delta_0 \times \Delta_0$ and $Q = \hat{F}^i_*(P|\Gamma)$*

$$\left|\frac{dQ}{dm}(x) \Big/ \frac{dQ}{dm}(y)\right| \leq C_*.$$

Proof Let $x_0, y_0 \in \Gamma$ be such that $\hat{F}^i(x_0) = x$ and $\hat{F}^i(y_0) = y$. Recall that $\hat{s}(x_0, y_0) \geq i$. Using Lemma A.5 and Lemma A.4 we obtain

$$\left|\frac{dQ}{dm}(x) \Big/ \frac{dQ}{dm}(y)\right| = \left|\frac{\Phi(x_0)}{J\hat{F}^i(x_0)} \cdot \frac{J\hat{F}^i(y_0)}{\Phi(y_0)}\right| \leq \frac{\Phi(x_0)}{\Phi(y_0)} \cdot \left|\frac{J\hat{F}^i(y_0)}{J\hat{F}^i(x_0)}\right| \leq \exp(C_\Phi + 2C_1).$$

We just have to take $C_* = \exp(C_\varphi + C_{\varphi'} + 2C_1)$. ☐

Now we are going to define a sequence of densities $\hat{\Phi}_0 \geq \hat{\Phi}_1 \geq \hat{\Phi}_2 \geq \cdots$ in $\Delta \times \Delta$ with the property that for all $i \geq 0$ and all $\hat{\Gamma} \in \hat{\xi}_i$

$$\pi_* \hat{F}^i_*\big((\hat{\Phi}_{i-1} - \hat{\Phi}_i)((m \times m)|\hat{\Gamma})\big) = \pi'_* \hat{F}^i_*\big((\hat{\Phi}_{i-1} - \hat{\Phi}_i)((m \times m)|\hat{\Gamma})\big). \tag{41}$$

Let $\varepsilon = \varepsilon(F) > 0$ be a small number to be determined later (see Lemma A.7 below). Let $i_1 = i_1(\Phi)$ be such that

$$C_\Phi \beta^{i_1} < C_{\hat{F}}. \tag{42}$$

For $i < i_1$, we take $\hat{\Phi} \equiv \Phi$. For $i \geq i_1$, let

$$\hat{\Phi}_i(z) = \left[\frac{\hat{\Phi}_{i-1}(z)}{J\hat{F}^i(z)} - \varepsilon \min_{w \in \hat{\xi}_i(z)} \frac{\hat{\Phi}_{i-1}(w)}{J\hat{F}^i(w)} \right] J\hat{F}^i(z), \tag{43}$$

where $\hat{\xi}_i(z)$ is the element of $\hat{\Gamma}$ of $\hat{\xi}_i$ that contains z. One can easily see that the sequence $\{\hat{\Phi}_i\}$ satisfies condition (41). The next result is proved in [11, Lemma 3]. As observed in [11, p. 166], ε depends only on β.

Lemma A.7 *If $\varepsilon > 0$ is sufficiently small, then there is $0 < \varepsilon_1 < 1$ (not depending on Φ) such that $\hat{\Phi}_i \leq (1 - \varepsilon_1)\hat{\Phi}_{i-1}$ for all $i \geq i_1$.*

A.3.3 Proof of (E_3)

Let $\varepsilon_1 > 0$ be as in Lemma A.7. Let $\Phi_0, \Phi_1, \Phi_2, \ldots$ be defined in the following way: given $n \geq 0$ and $z \in \Delta \times \Delta$, let

$$\Phi_n(z) = \hat{\Phi}_i(z) \quad \text{for } T_i(z) \leq n < T_{i+1}(z). \tag{44}$$

We claim that

$$\left| F_*^n \lambda - F_*^n \lambda' \right| \leq 2 \int \Phi_n \, d(m \times m) \quad \text{for all } n \geq 1. \tag{45}$$

Actually, taking $\Phi = \Phi_n + \sum_{k=1}^{n} (\Phi_{k-1} - \Phi_k)$ we have

$$\begin{aligned} \left| F_*^n \lambda - F_*^n \lambda' \right| &= \left| \pi_*(F \times F)_*^n (\Phi(m \times m)) - \pi'_*(F \times F)_*^n (\Phi(m \times m)) \right| \\ &\leq \left| \pi_*(F \times F)_*^n (\Phi_n(m \times m)) - \pi'_*(F \times F)_*^n (\Phi_n(m \times m)) \right| \\ &\quad + \sum_{k=1}^{n} \left| (\pi - \pi')_* \left[(F \times F)_*^n ((\Phi_{k-1} - \Phi_k)(m \times m)) \right] \right|. \end{aligned}$$

For the first term in the last sum we have

$$\left| \pi_*(F \times F)_*^n (\Phi(m \times m)) - \pi'_*(F \times F)_*^n (\Phi(m \times m)) \right| \leq 2 \int \Phi_n \, d(m \times m).$$

Let us see that all the other terms vanish. Define $A_{k,i} = \{z \in \Delta \times \Delta : k = T_i(z)\}$ and $A_k = \bigcup A_{k,i}$. Each $A_{k,i}$ is a union of elements of $\Gamma \in \hat{\xi}_i$ and $A_{k,i} \neq A_{k,j}$ for $i \neq j$. By (44) we have $\Phi_{k-1} - \Phi_k = \hat{\Phi}_{i-1} - \hat{\Phi}_i$ on $\Gamma \in \hat{\xi}_i | A_{k,i}$, and $\Phi_k = \Phi_{k-1}$ on $\Delta \times \Delta - A_k$. For $k \geq 1$

$$\begin{aligned} &\pi_*(F \times F)_*^n ((\Phi_{k-1} - \Phi_k)(m \times m)) \\ &\quad = \sum_i \sum_{\Gamma \subset A_{k,i}} F_*^{n-k} \pi_*(F \times F)_*^{T_i} ((\hat{\Phi}_{i-1} - \hat{\Phi}_i)(m \times m) | \Gamma) \end{aligned}$$

$$= \sum_i \sum_{\Gamma \subset A_{k,i}} F_*^{n-k} \pi'_* (F \times F)_*^{T_i} ((\hat{\Phi}_{i-1} - \hat{\Phi}_i)(m \times m)|\Gamma) \quad \text{by (41)}$$

$$= \pi'_* (F \times F)_*^n ((\Phi_{k-1} - \Phi_k)(m \times m)).$$

This completes the proof of (45). To finish (E_3) we write

$$\int \Phi_n d(m \times m) = \int_{\{T_{i_1} > n\}} \Phi_n d(m \times m) + \sum_{i=i_1}^{\infty} \int_{\{T_i \le n < T_{i+1}\}} \Phi_n d(m \times m).$$

Observe that

$$\int_{\{T_{i_1} > n\}} \Phi_n d(m \times m) = \int_{\{T_{i_1} > n\}} \Phi d(m \times m) = P\{T_{i_1} > n\},$$

while for $i \ge i_i$,

$$\begin{aligned}\int_{\{T_i \le n < T_{i+1}\}} \Phi_n d(m \times m) &= \int_{\{T_i \le n < T_{i+1}\}} \hat{\Phi}_i d(m \times m) \\ &\le \int_{\{T_i \le n < T_{i+1}\}} (1 - \varepsilon_1)^{i - i_1 + 1} \Phi d(m \times m) \\ &= (1 - \varepsilon_1)^{i - i_1 + 1} P\{T_i \le n < T_{i+1}\}.\end{aligned}$$

Hence

$$\begin{aligned}\left|F_*^n \lambda - F_*^n \lambda'\right| &\le 2P\{T_{i_1} > n\} + 2 \sum_{i=i_1}^{\infty} (1 - \varepsilon_1)^{i - i_1 + 1} P\{T_i \le n < T_{i+1}\} \\ &\le 2P\{T_{i_1} > n\} + 2(1 - \varepsilon_1)^{-i_1 + 1} \sum_{i=i_1}^{\infty} (1 - \varepsilon_1)^i P\{T_i \le n < T_{i+1}\}. \qquad (46)\end{aligned}$$

We may write

$$\begin{aligned}P\{T_{i_1} > n\} &= P\{T > n\} + (1 - \varepsilon_1)^{-i_1 + 1} \sum_{i=1}^{i_1 - 1} (1 - \varepsilon_1)^{i_1 - 1} P\{T_i \le n < T_{i+1}\} \\ &\le P\{T > n\} + (1 - \varepsilon_1)^{-i_1 + 1} \sum_{i=1}^{i_1 - 1} (1 - \varepsilon_1)^i P\{T_i \le n < T_{i+1}\},\end{aligned}$$

which together with (46) yields

$$\left|F_*^n \lambda - F_*^n \lambda'\right| \le 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} (1 - \varepsilon_1)^i P\{T_i \le n < T_{i+1}\},$$

with K_1 depending only on ε_1 and i_1. From (42) and Lemma A.7 one easily obtains the desired dependence of K_1 on φ and φ'.

A.3.4 Proof of (E_4)

This estimate is obviously true for $i = 0$. Take an arbitrary $i \geq 1$ and $\Gamma \in \hat{\xi}_i$. Recall that $\hat{F}^i$ maps $\hat{\Gamma}$ bijectively to $\Delta_0 \times \Delta_0$. Letting $Q = F_*^i(P|\Gamma)$ and observing that from (25) and (40) we have $T_{i+1} - T_i = T \circ \hat{F}^i$, we may write

$$P\{T_{i+1} - T_i > n \mid \Gamma\} = \frac{Q\{T > n\}}{Q(\Delta_0 \times \Delta_0)}.$$

Using Lemma A.6,

$$P\{T_{i+1} - T_i > n \mid \Gamma\} \leq C_*^2 \frac{(m \times m)\{T > n\}}{(m \times m)(\Delta_0 \times \Delta_0)}.$$

From this last inequality one easily obtains (E_4) with $K_2 = C_*^2/(m \times m)(\Delta_0 \times \Delta_0)$.

References

1. Alves, J.F., Bonatti, C., Viana, M.: SRB measures for partially hyperbolic systems whose central direction is mostly expanding. Invent. Math. **140**, 351–398 (2000)
2. Benedicks, M., Young, L.-S.: Markov extensions and decay of correlations for certain Hénon maps. Astérisque **261**, 13–56 (2000)
3. Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Mathematics, vol. 470. Springer, New York (1975)
4. Gouëzel, S.: Decay of correlations for nonuniformly expanding systems. Bull. Soc. Math. France **134**(1), 1–31 (2006)
5. Hofbauer, F., Keller, G.: Ergodic properties of invariant measures for piecewise monotonic transformations. Math. Z. **180**, 119–140 (1982)
6. Pianigiani, G.: First return maps and invariant measure. Israel J. Math. **35**, 32–48 (1980)
7. Ruelle, D.: A measure associated with Axiom A attractors. Am. J. Math. **98**, 619–654 (1976)
8. Sinai, Ya.: Gibbs measure in ergodic theory. Russ. Math. Surv. **27**, 21–69 (1972)
9. Viana, M.: Multidimensional nonhyperbolic attractors. Publ. Math. IHES **85**, 69–96 (1997)
10. Young, L.-S.: Statistical properties of dynamical systems with some hyperbolicity. Ann. Math. **147**, 585–650 (1998)
11. Young, L.-S.: Recurrence times and rates of mixing. Israel J. Math. **110**, 153–188 (1999)